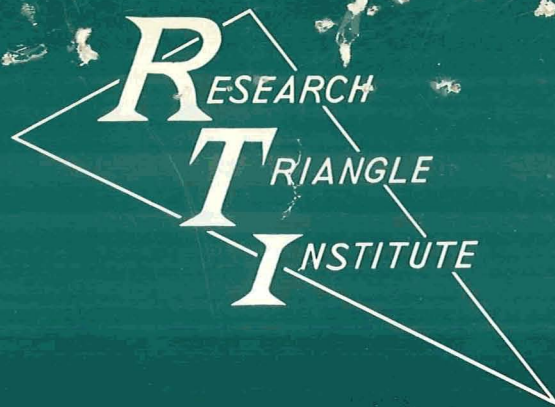




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M. R. Leadbetter

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I. INTRODUCTION

Much has been written concerning the mean number of crossings of a fixed level u , by a stationary Gaussian process in a given time T . Most recently (and most rigorously), this problem has been considered by Bulinskaya (1961) who derives the well known formula

$$(1) \quad E\{N(T)\} = \frac{1}{\pi} \left(\frac{-r''(0)}{r(0)} \right)^{1/2} e^{-u^2/(2r(0))}$$

under conditions which are very close to the necessary ones. Here,, $N(T)$ is the number of crossings of the level u in $(0,T)$, by the stationary Gaussian process $x(t)$ with covariance function $r(t)$.

This result is of considerable interest in a certain class of reliability problems where the process $x(t)$ is, for example, an error in system output and where it is desirable to keep $x(t)$ below a given level for the operating time of the system. Various ways in which (1) is useful in such situations have been discussed by Cramér (1962, a) and an application to a specific problem has been given by Leadbetter and Lewis (1962).

Briefly, for such applications, there are three ways in which it has been found profitable to use the formula (1) where we have a situation in which, for good system performance, $x(t)$ should be kept below the level u . First, we may regard the quantity $E\{N(T)\}$ itself as a system performance index and use it as a guide for deciding whether or not the system design is satisfactory from a reliability standpoint.

Alternatively, in cases where we are able to say that failure occurs if $x(t)$ ever crosses the level u , we may use (1) in obtaining upper bounds to the probability that failure occurs. A simple example of such a result is

$$p_0 \geq 1 - E\{N(T)\}$$

where p_0 is the probability of no crossing of the level u during $(0,T)$.

This formula is non-trivial only when $E\{N(T)\} < 1$ and is a sharp inequality in the case where $E\{N(T)\}$ is very small. However, the case when $E\{N(T)\}$ is very small is just that considered - or hoped for - in such reliability situations.

Finally, it is found that the formula (1) is useful in providing other indices of good system performance. For example, Cramér (1962b) has obtained a useful asymptotic result for maximum of a normal stationary process $x(t)$ in $0 \leq t \leq T$, and the formula (1) is useful in the derivation of this result.

Thus, from several points of view, the formula for the mean number of crossings of a level is important in certain reliability analyses. However, in such situations it may happen that larger errors in system output can be tolerated at certain times of the operating period than at others. For example, if $x(t)$ represents the velocity error of a missile system, it may be important to have $x(t)$ very small during certain critical periods. Thus, we are led to consider the corresponding problem when, instead of a fixed level u , we have a curve $u(t)$.

Further, in some cases of interest, we may require that the process $x(t)$ be non-stationary. One such situation - of particular application to a missile inertial guidance system - is that for which $x(t)$ is the integral of a normal stationary process. Cramér (1963) has obtained an expression corresponding to (1) in this case. Here, again it is sometimes of interest to consider crossings of an arbitrary curve rather than just a fixed level.

In each of these two situations, we are, in effect, considering the mean number of zero crossings of a non-stationary Gaussian process $x(t)-u(t)$, writing $u(t)$ for the arbitrary curve. We could formulate a general result for non-stationary Gaussian processes, from which the results for these cases would follow. However, the two cases are sufficiently distinct in detail to make separate treatments desirable, and accordingly will be treated separately. Moreover, it is obvious how a general result of this type should be formulated.

The method of proof of (1), given by Bulinskaya for the stationary case with fixed level, is essentially a rigorization of the derivation due to Kac (1943) and described by Grenander and Rosenblatt (1957). To avoid obscuring the essential simplicity of the ideas involved, we shall in Section 2 - for the curve crossing situation - give a heuristic argument which is along the same lines as that due to Kac for crossings of a fixed level.

In Section 3, we shall give the corresponding derivation for the integral of a stationary Gaussian process. In Section 4, the discussion of Section 2 will be made rigorous by following, essentially, the lines of proof given by Bulinskaya in the fixed level case. Finally, the modifications necessary to deal with the corresponding result for the integral of a stationary normal process will be indicated in Section 5.

2. Crossings of a curve $u(t)$ by a stationary Gaussian process (Heuristic Method)

Let $x(t)$ be a stationary Gaussian process with zero mean and covariance function $r(\tau) = E\{x(t)x(t+\tau)\}$ and (integrated) spectrum $F(\lambda)$. That is

$$r(\tau) = \int_0^{\infty} \cos \tau\lambda \, dF(\lambda) \dots$$

Define

$$\sigma_0^2 = \int_0^{\infty} dF(\lambda) = r(0)$$

$$\sigma_2^2 = \int_0^{\infty} \lambda^2 \, dF(\lambda) = -r''(0)$$

For $\epsilon > 0$ let the function $g_{\epsilon}(y)$ be defined by

$$(2) \quad \begin{aligned} g_{\epsilon}(y) &= 1 \quad \text{if } |y| \leq \epsilon \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Then $x(t)-u(t)$ is a normal process. Let $N(T)$ denote the number of crossings of $u(t)$ by $x(t)$ in $0 \leq t \leq T$, or, equivalently, the number of zero crossings by $x(t)-u(t)$. We then have, simply,

$$N(T) = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^T g_\epsilon [x(t)-u(t)] |x'(t)-u'(t)| dt$$

since, for sufficiently small ϵ , the contribution to the integral from each "up-crossing" or "down-crossing" is just 2ϵ . For the expectation of this quantity we obtain (on interchanging limits, expectations and integrations, with some abandon)

$$E\{N(T)\} = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^T E\{g_\epsilon [x(t)-u(t)] |x'(t)-u'(t)|\} dt.$$

Since for a stationary Gaussian process, $x(t)$ and $x'(t)$ are independent random variables for any given t , we may write this expression as

$$\begin{aligned} E\{N(T)\} &= \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^T E\{g_\epsilon [x(t)-u(t)]\} E\{|x'(t)-u'(t)|\} dt \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^T \left[\int_{u(t)-\epsilon}^{u(t)+\epsilon} \sigma_0^{-1} \phi(y/\sigma_0) dy \right] E\{|x'(t)-u'(t)|\} dt \end{aligned}$$

where $\phi(y)$ is the standard normal density function.

Thus

$$E\{N(T)\} = \sigma_0^{-1} \int_0^T \phi(u(t)/\sigma_0) E\{|x'(t)-u'(t)|\} dt.$$

Now for given t , $x'(t)$ is a normal random variable with zero mean and variance σ_2^2 . Hence,

$$\begin{aligned} E\{|x'(t)-u'(t)|\} &= \sigma_2^{-1} \int_{-\infty}^{\infty} |y-u'(t)| \phi(y/\sigma_2) dy \\ &= \sigma_2 \int_{-\infty}^{\infty} |y-u'(t)/\sigma_2| \phi(y) dy \\ &= 2\sigma_2 (2\pi)^{-1/2} \exp\{-(u'(t))^2/(2\sigma_2^2)\} + u'(t) [2\Phi(u'(t)/\sigma_2) - 1] \end{aligned}$$

writing $\Phi(y)$ for the standard normal distribution function.

Thus, finally

$$(3) \quad E\{N(T)\} = \sigma_0^{-1} \int_0^T \phi(u(t)/\sigma_0) [2\sigma_2 \phi(u'(t)/\sigma_2) + u'(t)\{2\Phi(u'(t)/\sigma_2) - 1\}] dt$$

Examples

(i) In the case where $u(t) = c$, a constant, independent of t , the second integral vanishes, and the first reduces to give the standard result

$$E\{N(T)\} = \frac{T}{\pi} \frac{\sigma_2}{\sigma_0} e^{-c^2/(2\sigma_0^2)}$$

(ii) For a "straight line barrier" with gradient $b \neq 0$, $u(t) = a + bt$,

$$\begin{aligned} E\{N(T)\} &= \sigma_0^{-1} [2\sigma_2 \phi(b/\sigma_2) + b(2\Phi(b/\sigma_2) - 1)] \int_0^T \phi[(a + bt)/\sigma_0] dt \\ &= [2 \frac{\sigma_2}{b} \phi(b/\sigma_2) + 2\Phi(b/\sigma_2) - 1] [\Phi((a + bT)/\sigma_0) - \Phi(a/\sigma_0)]. \end{aligned}$$

3. Curve-crossings by the integral of a stationary Gaussian process (Heuristic Method.)

Suppose now that $x(t)$ is a stationary Gaussian process with covariance function $r(\tau)$, spectrum $F(\lambda)$, and that $z(t) = \int_0^t x(s) ds$. Let $E\{x(t)\} = 0$ and for convenience suppose $\text{var}\{x(t)\} = 1$. If for some $b > 1$, $\int_0^\infty [\log(1+\lambda)]^b dF(\lambda) < \infty$ the sample functions of $x(t)$ will with probability one, be everywhere continuous. Cramér (1963) has shown that, under these conditions the following formulae are valid

$$\begin{aligned} E\{z(t)\} &= 0 \\ E\{z^2(t)\} &= \sigma^2(t) = \int_0^t \int_0^t r(u-v) du dv \\ &= 2 \int_0^\infty \frac{1 - \cos t\lambda}{\lambda^2} dF(\lambda) \\ (4) \quad E\{x(t)z(t)\} &= \sigma(t)\Psi(t) = \int_0^t r(u) du \\ &= \int_0^\infty \frac{\sin t\lambda}{\lambda} dF(\lambda), \end{aligned}$$

and that if $N_z(T)$ denotes the number of crossings of the fixed level c by $z(t)$ in $0 \leq t \leq T$,

$$E\{N_z(T)\} = \int_0^T Q(t) dt$$

where

$$Q(t) = \frac{1}{2\pi\sigma} e^{-c^2/(2\sigma^2)} [2(1-\Psi^2)^{1/2} \exp\{-\frac{c^2\Psi^2}{2\sigma^2(1-\Psi^2)}\} + \sqrt{2\pi} \frac{c\Psi}{\sigma} [1-2\Phi(\frac{-c\Psi}{\sigma\sqrt{1-\Psi^2}})]]$$

writing σ and Ψ for $\sigma(t)$, $\Psi(t)$, respectively.

Suppose now that we consider crossings by $z(t)$ of the curve $u(t)$. We shall show that the above formula is modified to

$$(5) \quad E\{N_z(T)\} = \int_0^T Q(t) dt$$

where now $Q(t)$ has the form, writing u for $u(t)$,

$$Q(t) = \sigma^{-1} (1-\Psi^2)^{1/2} \phi(u/\sigma) [2\phi(\eta) + \eta(2\Phi(\eta)-1)]$$

in which $\eta = \eta(t) = (u'(t)-u\Psi/\sigma)/(1-\Psi^2)^{1/2}$.

For we have, in the same way as previously,

$$\begin{aligned} E\{N_z(T)\} &= \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^T E\{g_\epsilon(z(t)-u(t)) |x(t)-u'(t)|\} dt \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^T dt \int_{u(t)-\epsilon}^{u(t)+\epsilon} ds \int_{-\infty}^{\infty} |v-u'(t)| f(s,v) dv \end{aligned}$$

where

$$f(s,v) = [2\pi\sigma(1-\Psi^2)^{1/2}]^{-1} \exp\{-[2(1-\Psi^2)]^{-1}[s^2/\sigma^2 - 2\Psi vs/\sigma + v^2]\}.$$

From this it follows that

$$\begin{aligned} E\{N_z(T)\} &= \int_0^T dt \int_{-\infty}^{\infty} |v-u'(t)| f(u(t),v) dv \\ &= \int_0^T \frac{dt}{2\pi\sigma} e^{-u^2/(2\sigma^2)} \int_{-\infty}^{\infty} |\omega(1-\Psi^2)^{1/2} + u\Psi/\sigma - u'| e^{-\omega^2/2} d\omega \end{aligned}$$

(where the quantities σ , Ψ , u , u' , are all functions of t).

Thus with η defined as above,

$$E\{N_z(T)\} = \int_0^T Q(t) dt, \text{ where}$$

$$Q(t) = (2\pi\sigma^2)^{-1/2} (1-\Psi^2)^{1/2} \phi(u/\sigma) \int_{-\infty}^{\infty} |\omega - \eta| e^{-\omega^2/2} d\omega$$

which easily reduces to the form required in (5).

4. Rigorous derivation of equation (3)

Throughout this section, we consider a normal stationary process $x(t)$ with zero mean and such that its spectrum $F(\lambda)$ satisfies the condition

$$(6) \quad \int_0^{\infty} \lambda^2 [\log(1+\lambda)]^b dF(\lambda) < \infty \quad \text{for some } b > 1.$$

It then follows from Hunt (1951) that, with probability one, the sample functions of the process has a derivative $x'(t)$ which is everywhere continuous. A statement of Hunt's result - in a convenient form for our purposes - has been given by Belaev (1961). Since the curve $u(t)$ is assumed to have a continuous derivative it follows that the normal process $x_*(t) = x(t) - u(t)$ has a continuous derivative with probability one. Further, since the one-dimensional distributions of $x_*(t)$ are bounded in $0 \leq t \leq T$ (being univariate normal distributions), it follows from theorem 1 of Bulinskaya (1961), that the events that $x_*(t)$ should touch the axis anywhere or cross it an infinite number of times in $0 \leq t \leq T$ both have probability zero. The same is therefore true of crossings the curve $u(t)$ by the process $x(t)$.

The derivation to be given will follow closely the pattern of that by Bulinskaya. In fact, the construction of a sequence of auxiliary processes and the derivation to equation (7) will be precisely as given by her in the fixed level case. From there the proof differs in detail and in some minor simplifications. Stated formally, then, we have the following result.

Theorem

Let $x(t)$ be a separable normal stationary process for which (6) holds. Let $u(t)$ have a continuous first derivative $u'(t)$, $0 \leq t \leq T$ and let $N_x(T)$ denote the number of crossings of $u(t)$ by $x(t)$ in $0 \leq t \leq T$.

Then, with the previous notation, equation (3) holds.

Proof:

There is no loss in generality in taking $T = 1$. For fixed t , $0 \leq t \leq 1$ write $k_n = k_n(t)$, for the unique integer such that $\frac{k_n}{2^n} \leq t < \frac{k_n+1}{2^n}$ ($k_n = 0, 1, \dots, 2^n$). Define for each n , a process $\{y_n(t)\}$ as follows:

$$y_n(t) = x_*(k_n/2^n) + 2^n [x_*(k_n+1)/2^n - x_*(k_n/2^n)] (t - k_n/2^n).$$

That is, the $y_n(t)$ -process consists of a series of straight lines with vertices at points $\frac{1}{2^n}$ apart.

Write N_x for $N_x(1)$, the number of crossings of the curve $u(t)$ by $x(t)$ in $0 \leq t \leq 1$, and N_{y_n} for the number of zero crossings of $y_n(t)$ in that time. Then since N_x is also the number of zero crossings of $x_*(t)$ and since in any interval $(\frac{k}{2^n}, \frac{k+1}{2^n})$, $y_n(t)$ may cross the axis at most once it follows that $N_{y_n} \leq N_x$. Further, it is clear that with probability one, $N_{y_n} \rightarrow N_x$, since N_x is finite with probability one. Thus, it follows from the theorem concerning the convergence of the integrals of increasing sequences of non-negative functions, that $E\{N_{y_n}\} \rightarrow E\{N_x\}$ as $n \rightarrow \infty$.

Using the same notation as before it is clear that

$$N_{y_n} = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^1 g_\epsilon(y_n(t)) |y_n'(t)| dt.$$

$$\text{But } \frac{1}{2\epsilon} \int_0^1 g_\epsilon(y_n(t)) |y'_n(t)| dt = \sum_{k=0}^{2^n-1} \frac{1}{2\epsilon} \int_{k/2^n}^{(k+1)/2^n} g_\epsilon(y_n(t)) |y'_n(t)| dt$$

$$\leq 2^n \dots \text{since each integral is}$$

dominated by 2ϵ . Hence, by dominated convergence the order of limit and expectation signs may be inverted to give

$$\begin{aligned} E\{N_{y_n}\} &= \lim_{\epsilon \rightarrow 0} \left[\frac{1}{2\epsilon} E\left\{ \int_0^1 g_\epsilon(y_n(t)) |y'_n(t)| dt \right\} \right] \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^1 E\{g_\epsilon(y_n(t)) |y'_n(t)| dt\}, \text{ using Fubini's theorem} \end{aligned}$$

for positive functions.

Thus we have

$$(7) \quad E\{N_{y_n}\} = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^1 \int_{-\infty}^{\infty} \int_{-\epsilon}^{\epsilon} |y| P_n(x, y) dx dy dt$$

which is identical with Bulinskaya's formula (4) except that u (written for $u(t)$) is now time dependent and $P_n(x, y)$ has here the following definition

$$P_n(x, y) = \frac{1}{2\pi\sqrt{D}} \exp\left[-\frac{1}{2D}\{C(x-\alpha)^2 - 2B(x-\alpha)(y-\beta) + A(y-\beta)^2\}\right]$$

where

$$\alpha = \alpha_n(t) = E\{y_n(t)\}$$

$$\beta = \beta_n(t) = E\{y'_n(t)\}$$

$$A = A_n(t) = \text{var}\{y_n(t)\}$$

$$C = C_n(t) = \text{var}\{y'_n(t)\}$$

$$B = B_n(t) = \text{cov}\{y_n(t), y'_n(t)\}$$

and

$$D = D_n(t) = AC - B^2.$$

From the definition of $y_n(t)$,

$$- \alpha = u(k_n/2^n) + 2^n \{u[(k_n+1)/2^n] - u(k_n/2^n)\} (t - k_n/2^n)$$

$$- \beta = 2^n [u[(k_n+1)/2^n] - u(k_n/2^n)]$$

whereas the variances and covariances have the same form as Bulinskaya's, viz,

$$A = r(0) [(1-2^n t + k_n)^2 + (2^n t - k_n)^2] + 2r(1/2^n)(2^n t - k_n)(1-2^n t + k_n)$$

$$B = 2^n [(2^n t - k_n) - (1-2^n t + k_n)] [r(0) - r(1/2^n)]$$

$$C = 2^{2n+1} [r(0) - r(1/2^n)].$$

Writing $r(2^{-n}) = r(0) + 2^{-(2n+1)} r''(\theta_n)$, $0 < \theta_n < 2^{-n}$, ($r'(0) = 0$), and noting that $t - k_n/2^n \leq 2^{-n}$, $(k_n+1)/2^n - t \leq 2^{-n}$, we have,

$$A = r(0) + 2^{-2n} r''(\theta_n) (2^n t - k_n) (1-2^n t + k_n)$$

$$|A - r(0)| \leq |r''(\theta_n)| 2^{-2n} \rightarrow 0.$$

Hence, $A = A_n(t) \rightarrow r(0)$ as $n \rightarrow \infty$, uniformly in t .

$$B = -[(t - k_n/2^n) - ((k_n+1)/2^n - t)] r''(\theta_n)/2.$$

Thus $|B| \leq 2^{-n} |r''(\theta_n)|$.

Hence, $B = B_n(t) \rightarrow 0$ as $n \rightarrow \infty$, uniformly in t .

$$C = r''(\theta_n) \longrightarrow -r''(0) \text{ as } n \longrightarrow \infty.$$

Thus, $C = C_n(t) \longrightarrow -r''(0)$ as $n \longrightarrow \infty$, uniformly in t . ($C_n(t)$ is actually independent of t).

Further,

$$\begin{aligned} u(t) + \alpha &= u(t) + \alpha_n(t) \\ &= u(t) - u(k_n/2^n) - 2^n(t - k_n/2^n)[u((k_n+1)/2^n) - u(k_n/2^n)]. \end{aligned}$$

Now

$$u[(k_n+1)/2^n] = u(k_n/2^n) + 2^{-n} u'(k_n/2^n + \lambda_n), \quad 0 < \lambda_n < 2^{-n}.$$

Hence,

$$\begin{aligned} 2^n\{u[(k_n+1)/2^n] - u(k_n/2^n)\} &= u'(k_n/2^n + \lambda_n) \\ &\longrightarrow u'(t) \text{ as } n \longrightarrow \infty. \end{aligned}$$

From this it follows that

$$u(t) + \alpha \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

$$\text{i.e.,} \quad \alpha = \alpha_n(t) \longrightarrow -u(t) \text{ as } n \longrightarrow \infty.$$

Also,

$$\begin{aligned} -\beta &= 2^n [u((k_n+1)/2^n) - u(k_n/2^n)] \\ &\longrightarrow u'(t), \text{ as above, when } n \longrightarrow \infty. \end{aligned}$$

It is clear that the limits for $\alpha_n(t)$, $\beta_n(t)$ are also uniform in t since $u'(t)$ is uniformly continuous in $0 \leq t \leq 1$.

Thus, to summarize, we have the following uniform limits in $0 \leq t \leq 1$,

$$\begin{aligned} (8) \quad A_n(t) &\longrightarrow r(0) \\ B_n(t) &\longrightarrow 0 \\ C_n(t) &\longrightarrow -r''(0) \\ D_n(t) &\longrightarrow -r(0)r''(0) \\ \alpha_n(t) &\longrightarrow -u(t) \\ \beta_n(t) &\longrightarrow -u'(t) \end{aligned}$$

We have that

$$\begin{aligned} \int_{-\epsilon}^{\epsilon} P_n(x,y) dx &\leq 2\epsilon \sup_x P_n(x,y) \\ &= \frac{\epsilon}{\pi \sqrt{D}} \exp [-(y-\beta)^2/(2C)] \end{aligned}$$

and
$$\int_{-\infty}^{\infty} |y| \exp [-(y-\beta)^2/(2C)] dy < \infty.$$

Hence, from (7), by dominated convergence,

$$\begin{aligned} E\{N_{y_n}\} &= \int_0^1 dt \int_{-\infty}^{\infty} dy \left\{ \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} |y| P_n(x,y) dx \right\} \\ &= \int_0^1 dt \int_{-\infty}^{\infty} |y| P_n(0,y) dy \\ &= \frac{1}{2\pi \sqrt{D}} \int_0^1 e^{-\alpha^2/(2A)} dt \int_0^{\infty} y \{ \exp[-\frac{A}{2D} (y-\gamma\sqrt{\frac{D}{A}})^2] \\ &\quad + \exp[-\frac{A}{2D} (y+\gamma\sqrt{\frac{D}{A}})^2] \} dy \end{aligned}$$

where $\gamma = \gamma_n(t) = \sqrt{\frac{A}{D}} (\beta - \frac{B}{A} \alpha).$

Thus,

$$\begin{aligned} E\{N_{y_n}\} &= (2\pi A)^{-1} D^{1/2} \int_0^1 e^{-\alpha^2/(2A)} dt \int_0^{\infty} y (e^{-(y-\gamma)^2/2} + e^{-(y+\gamma)^2/2}) dy \\ &= (2\pi A)^{-1} D^{1/2} \int_0^1 e^{-\alpha^2/(2A)} \{ 2e^{-\gamma^2/2} + \gamma(2\pi)^{1/2} [\Phi(\gamma) - \Phi(-\gamma)] \} dt \\ &= A^{-1} (2D/\pi)^{1/2} \int_0^1 e^{-\alpha^2/(2A)} [\phi(\gamma) + \gamma(\Phi(\gamma) - 1/2)] dt. \end{aligned}$$

From (8) it follows that

$$\gamma = \gamma_n(t) \longrightarrow -u'(t)/(-r''(0))^{1/2}$$

and since the convergence is uniform, $\gamma_n(t)$ is uniformly bounded.

Hence, by bounded convergence, using (8),

$$E\{N_{y_n}\} \longrightarrow [-r''(0)/r(0)]^{1/2} \int_0^1 \phi[u(t)/(r(0))^{1/2}] \{2\phi[-u'(t)/(-r''(0))^{1/2}] - [u'(t)/(-r''(0))^{1/2}][2\phi\{-u'(t)/(-r''(0))^{1/2}\} - 1]\} dt.$$

Equation (3) now follows, on putting $r(0) = \sigma_0^2$, $-r''(0) = \sigma_2^2$, and using the fact, already noted, that $E\{N_{y_n}\} \longrightarrow E\{N_x\}$ as $n \longrightarrow \infty$.

5. Justification of equation (5)

Let $\int_0^\infty [\log(1+\lambda)]^b dF(\lambda) < \infty$ for some $b > 1$. Then proceeding along the same lines as in section (4), we may obtain the equation corresponding to (7), viz,

$$(9) \quad E\{N_{y_n}\} = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^1 \int_{-\infty}^\infty \int_{-\epsilon}^\epsilon |y| P_n(x, y) dx dy dt$$

where now $y_n(t)$ is the process consisting of line segments as before, but derived from $z(t) - u(t)$, and

$$P_n(x, y) = \frac{1}{2\pi \sqrt{D}} \exp \left[-\frac{1}{2D} \{C(x-\alpha)^2 - 2B(x-\alpha)(y-\beta) + A(y-\beta)^2\} \right]$$

$$\alpha = \alpha_n(t) = E\{y_n(t)\}, \quad \beta = \beta_n(t) = E\{y'_n(t)\}$$

$$A = A_n(t) = \text{var}\{y_n(t)\}, \quad C = C_n(t) = \text{var}\{y'_n(t)\}$$

$$B = B_n(t) = \text{cov}\{y_n(t), y'_n(t)\}, \quad D = D_n(t) = AC - B^2.$$

α and β have the same forms as before and therefore have, again, the uniform limits $-u(t)$, $-u'(t)$, respectively, as $n \longrightarrow \infty$. We shall now investigate the limiting behavior of the quantities A , B , C . From the definition of the process $\{y_n(t)\}$ we obtain,

$$A_n(t) = \sigma^2(k_n/2^n) (1-2^{n-t+k_n})^2 + \sigma^2((k_n+1)/2^n) (2^{n-t-k_n})^2 \\ + (k_n+1-2^{n-t})(2^{n-t-k_n}) [\sigma^2(k_n/2^n) + \sigma^2((k_n+1)/2^n) - \sigma^2(1/2^n)]$$

where use has been made of the fact that

$$\text{cov}\{z_n(t), z_n(u)\} = \frac{1}{2} (\sigma^2(t) + \sigma^2(u) - \sigma^2(t-u)),$$

a formula given by Cramér (1963), and easily derivable from the definition of the $z_n(t)$ -process.

Hence,

$$\begin{aligned} (10) \quad A_n(t) &= \sigma^2(k_n/2^n)(k_n+1-2^n t) + \sigma^2((k_n+1)/2^n)(2^n t - k_n) \\ &\quad - \sigma^2(1/2^n)(k_n+1-2^n t)(2^n t - k_n). \end{aligned}$$

Now, from (4), we have

$$\sigma^2(t) = t^2 \int_0^\infty \frac{1 - \cos t\lambda}{(t\lambda)^2/2} dF(\lambda)$$

and it is thus clear from dominated convergence that

$$(11) \quad \sigma^2(t) \sim t^2 \text{ as } t \rightarrow 0.$$

The third term on the right of (10) is therefore dominated by $\sigma^2(1/2^n)$, since $k_n \leq 2^n t < k_n + 1$, and this tends to zero (uniformly in t) as $n \rightarrow \infty$.

It is also clear from (4) that $\sigma^2(t)$ has the continuous derivative $2\sigma(t)\Psi(t)$ and hence we may write

$$\begin{aligned} (12) \quad \sigma^2((k_n+1)/2^n) &= \sigma^2(t) + 2[(k_n+1)/2^n - t] \sigma(t_1)\Psi(t_1) \\ \sigma^2(k_n/2^n) &= \sigma^2(t) - 2(t - k_n/2^n) \sigma(t_2)\Psi(t_2) \end{aligned}$$

$$\text{where } |t_1 - t| < 2^{-n}, \quad |t_2 - t| < 2^{-n}.$$

Hence, the first two terms on the right of (10) may be written as

$$\sigma^2(t) + E_n(t)$$

where for some constant K , $|E_n(t)| \leq K/2^n$ for $0 \leq t \leq 1$.

From this it follows that

$$A_n(t) \longrightarrow \sigma^2(t), \text{ uniformly in } 0 \leq t \leq 1, \text{ as } n \longrightarrow \infty.$$

Correspondingly we have, for B,

$$\begin{aligned} B_n(t) &= 2^n \{ \sigma^2((k_n+1)/2^n) (2^n t - k_n) - \sigma^2(k_n/2^n) (k_n+1-2^n t) \\ &\quad + \frac{1}{2} [k_n+1-2^n t - (2^n t - k_n)] [\sigma^2(k_n/2^n) + \sigma^2((k_n+1)/2^n) - \sigma^2(1/2^n)] \} \\ (13) \quad &= 2^{n-1} [\sigma^2((k_n+1)/2^n) - \sigma^2(k_n/2^n)] - 2^{n-1} \sigma^2(1/2^n) [1-2(2^n t - k_n)]. \end{aligned}$$

From (12), and the (uniform) continuity of $\sigma(t)\Psi(t)$, it follows that the first term in (13) converges uniformly to $\sigma(t)\Psi(t)$ as $n \longrightarrow \infty$. The second term in (13) is

$$2^{2n} \sigma^2(2^{-n}) [(k_n+1)/2^n - t - (t - k_n/2^n)] / 2$$

which tends to zero (uniformly) as $n \longrightarrow \infty$, by (11).

Hence,

$$B_n(t) \longrightarrow \sigma(t)\Psi(t), \text{ uniformly in } 0 \leq t \leq 1, \text{ as } n \longrightarrow \infty.$$

Finally, for C, we have,

$$C_n(t) = 2^{2n} \sigma^2(1/2^n)$$

$$\longrightarrow 1 \text{ as } n \longrightarrow \infty, \text{ by (11).}$$

With these uniform limits for A, B, C, α , β , (and hence the limit $\sigma^2(t)(1-\Psi^2(t))$ for D) we may now proceed exactly as in section 4. Thus, equation (5) follows from (9).

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